SPACE AND ITS MEASURES

CHAPTER 3

We began the last chapter by describing time as "one of the basic notions of science," and went on to speak at some length about time. Perhaps you noticed that we were unable to confine the discussion to time and time alone. We were speaking also of positions and distances, of motion, of matter.

These, too, are basic notions. Each of them is intertwined with all the others. It is impossible to deal with one without dealing with all the rest. To discuss them intelligibly, it is necessary to deal with them one after another, although they do not appear in nature one after another. They come together.

To use these notions skillfully, we must refine our understanding of each of them, and we must do this even though, strictly speaking, we have no place to start. The procedure — and all our history has shown it to be a very practical procedure — is to move back and forth among these basic notions, registering gains wherever we can and using these gains in turn to register further gains. We use our crude notions of space, for example, to refine our notion of time. Then we are able to use our refined knowledge of time to improve our notion of space. But while we carry on this process, we must remember that physics itself is one indivisible subject dealing with the whole universe of which we are part. We subdivide it for our own convenience, but only so that later we may put it together again.

What we can say with assurance is that size and distance, as divisions of space, determine the nature of the world just as much as does time. Think of the sun, whose bright glare showers the earth with light, makes our crops grow, and keeps our earth from being a dead, frozen planet. It appears to us as a large, bright disc, too brilliant to look at directly. Compare it with a star which appears as nothing but a tiny bright spark in the dark night. The difference is one of distance. The sun is a star we happen to live "near." The little twinkling star, though a furiously hot sun like our own, is relatively far away from us.

Or consider the very small. You know that a drop of pond water looks a little cloudy perhaps to the unaided eye, but nothing more. Under the microscope, it is a jungle of plants and animals, living, hunting, fleeing. Beyond the microscope's grasp is a still more wonderful part of the world, the world of the atom, which we are going to probe.

When we say the sun is "near," compared to any other star, or that Bombay is "far," compared to any place in our state, we have started to measure intervals of distance, or size. Intervals of space — sizes or distances — can best be compared by the same scheme we used in comparing time intervals. We have only to make a count. We count how many times we have to span with our fingertips, or lay off with a ruler, or pace out with equal steps, and we have measured a distance.

3–1. The Unit of Distance

Every people has had a unit of length. Hunting folk, like the North American Indians, used the pace, the bowshot, and the day's journey. When it became necessary to measure off land for irrigation and for plowing, standard rods were made. As early as ancient Egyptian times, when great buildings were made of stone, rather wide use was made of a standard of length, a cubit, or the distance from the elbow to the tip of the middle finger. In the times of the Ptolemies, there were professional pacers who helped make maps by pacing out the roads in units called stadia. By medieval times, with the growth of the European nations, there were many measuring units. In England, length was measured by the inch, foot, yard, fathom, rod, furlong, mile, and league. These units reflect convenient early standards.

The French Revolution brought to power a government set sharply against all that was traditional and old-fashioned. An early action of the new government of France was the establishment of a group of learned men ("experts," we would say nowadays) to produce a rational set of units for all measurements, the common, everyday ones as well as those of science and the blossoming technologies. They set up standards of length, among many others, which have become world-wide in science, and nearly world-wide in everyday life. They called their scheme of units the metric system, and its fundamental length unit is the meter (from the Greek *metron,* to measure). They felt it was better to adopt a length standard which had some more lasting significance than the length of the pace, and they were convinced of the value of the decimal system. They therefore chose the meter to be one ten-millionth (10^{-7}) of the distance from the equator to the North Pole. In the 1790's, this dimension was rather well known in terms of carefully laid out base lines surveyed in Europe. This is the origin of the metric system, which we employ throughout physics in all countries today.

It was one thing to say that the meter was to be 10^{-7} of a quadrant of the earth's circumference, but quite another to lay off this distance on a short metal bar. However, it is not important that the standard meter be related to the earth's circumference. As our standard of length we now employ the standard meter bar the French made. Many careful copies have been produced.

The Founding Fathers of our own American republic, very much steeped in the same climate of opinion that later produced the metric system of weights and measures in France, introduced a kind of "metric system" in currency which we use to this day. They established our decimal system of coinage, with 100 cents = 1 dollar, and a few other multiples, like dimes and quarters, to replace the traditional English system in which 12 pence = 1 shilling, and 20 shillings = 1 pound sterling. Anyone who struggles to calculate a 10 per cent discount on the price of an English book will see the virtue of a system of units which matches the number system. This is the great value of the metric system, which has made it universal in science. Even our inch is now legally defined in terms of the meter: it is defined as exactly 2.54×10^{-2} meters.

It is worth remembering that a meter is roughly a yard and a foot is about 30 cm, while a millimeter is about the thickness of a pencil lead.

Names and Definitions of Metric Units of Distance

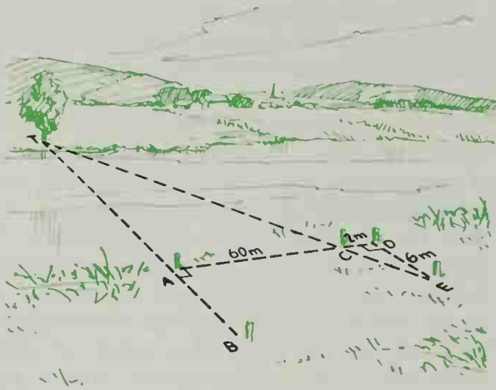
1 kilometer (km)	= 10^3 meters (m)	
1 centimeter (cm)	= 10^{-2} m	
1 millimeter (mm)	= 10^{-3} m	= 10^{-1} cm
1 micron (μ)	= 10^{-6} m	= 10^{-3} mm
1 Angstrom (A)	= 10^{-10} m	= 10^{-8} cm

Note that in the table the prefix *kilo* means 10^3, *centi* is 10^{-2}, *milli* is 10^{-3}, and *micro* is 10^{-6}. Another prefix, often used, is *mega*, which means 10^6. A widespread habit has grown up among American physicists in recent years of referring to a large sum of money as a "megabuck." This use of Greek and Latin prefixes for multiples and submultiples of a unit has been extended to many different units. Have you ever heard of a *megohm* or a *microfarad?* What is a *millisecond?*

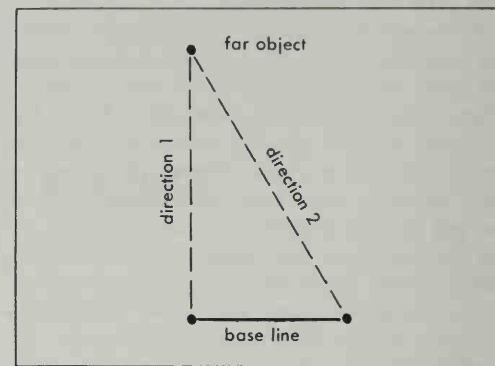
Throughout this book, to save time and space, we shall follow the practice of physicists and, whenever convenient, use the abbreviations given in the table instead of writing out the full name of the unit.

3–2. Measuring Large Distances — Triangulation

The method of laying standard lengths end to end can be used to measure quite large distances on the surface of the earth. It is sometimes used



3–1. Measuring a distance by triangulation.



3–2. When we have measured the length of a base line and know the direction of an object from each end of the base line, we can find the distance to the object.

in surveying, but often it becomes inconvenient. To measure the distance across a river, or the height of a mountain, or indeed the distance to a star, we can use a simple, indirect method. This method is based on the geometry of a triangle and is called *triangulation.*

One way to measure a distance by triangulation is illustrated in Fig. 3–1. We wish to measure the distance AT across the river. To do so we line up the tree, T, on the far side of the river, with two stakes A and B. We then construct the right angle BAC. (To do this we can use a large carpenter's square.) We drive a stake into the ground at C, a measured distance from A. Then we continue along line AC and drive another stake, D, into the ground a measured distance farther on. Now we construct a right angle CDE. We drive in a stake at E so placed that it is in line with the tree, T, and the stake, C. Finally we measure the distance DE.

As you can see in the figure, the triangles TAC and EDC are similar because they have two pairs of equal angles, the right angles TAC and EDC and the vertical angles TCA and ECD. Therefore the corresponding sides are in the same ratio. In particular

$$\frac{AT}{DE} = \frac{AC}{DC},$$

and the distance AT across the river is

$$AT = \frac{AC}{DC} \times DE.$$

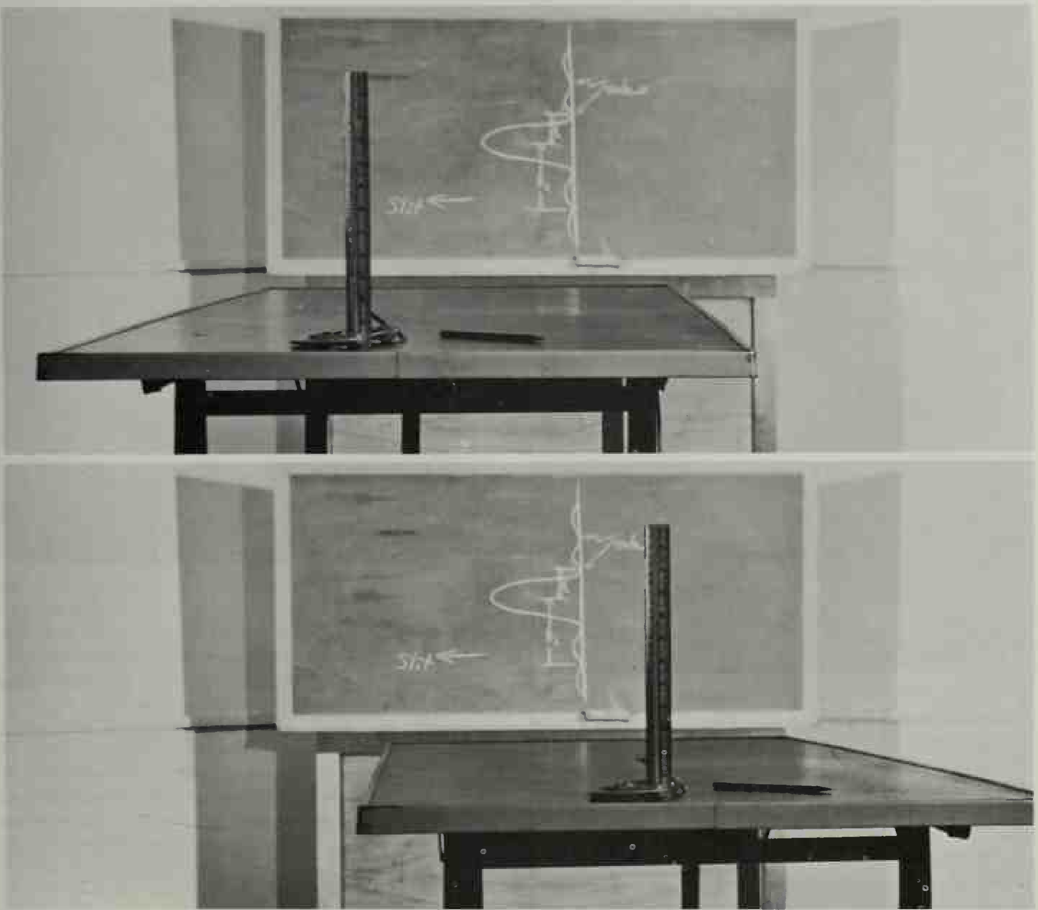
Since we have measured AC, DC, and DE, we can now determine the distance AT across the river. For example, if we suppose that the measured distances are $AC = 60$ m, $DC = 2$ m, and $DE = 6$ m, then the width of the river is

$$AT = \tfrac{60}{2} \times 6\text{m} = 180\text{m}.$$

We can simplify the procedure. All we need to do is to construct the right angle TAC; measure off a convenient distance, AC, called the base line; and measure the angle TCA. By making a scale drawing we can get the answer.

An instrument can be constructed that will measure the angles and work out the geometry automatically. An example of such a triangulating instrument, for measuring distances by simple sighting, is the range finder found in almost all good cameras. The base line of the range finder is no larger than the camera, and distant objects will appear at almost the same angle from both ends of it. Just where the sight lines cross is then difficult to say. The camera range finder, therefore, measures only the distances to near-by objects. You can make range finders with longer base lines, and you will find that the bigger you make the base line, the greater the distance you can measure.

The big range finders on warships have base lines limited by the size of the ship. To measure the distances of planets, astronomers use base lines extending over half the earth. The largest base line we can use is the diameter of the earth's orbit, the distance from one point on its path around the sun to the point reached a half year later. This sets the limit for measuring big distances geometrically.

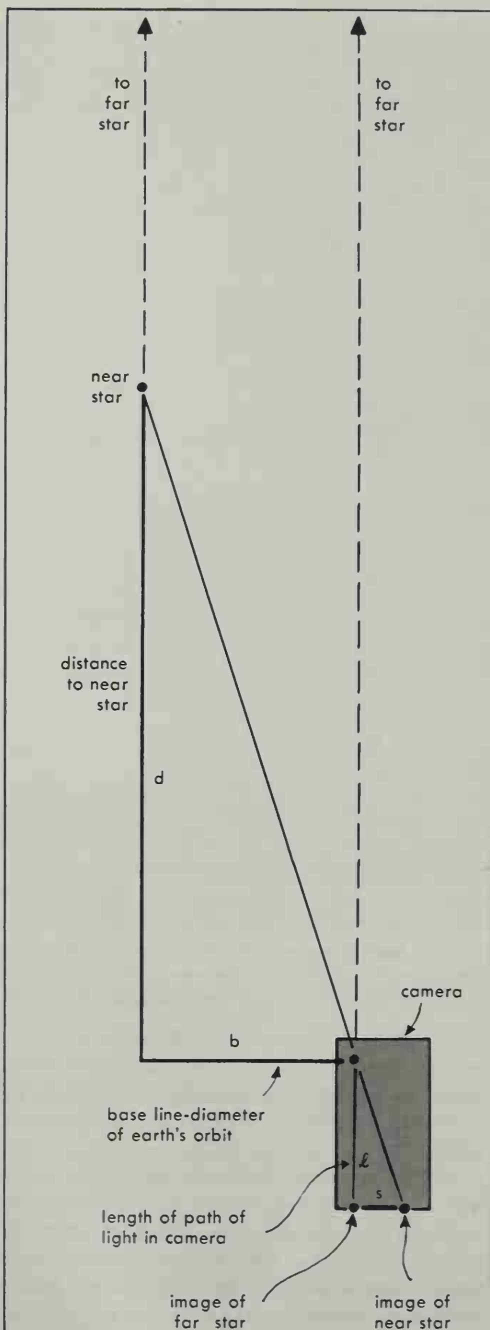


To use a base line to find the distance to an object, we must measure the direction of the object from each end of the base line. (Fig. 3–2.) To measure these two directions, an astronomer who uses the diameter of the earth's orbit as a base line must have a way of establishing a reference direction when he takes his observation at one end of the diameter and when he takes it at the other end. To fix this reference direction, he uses the most distant stars. He can pick them out because they do not change their apparent positions with respect to one another. Here he is using something very familiar to you. As you watch through a window of a rapidly moving car, the objects near you apparently move rapidly backward but the distant features of the landscape do not seem to move at all. In the same way, the most distant stars stand still relative to each

3–3. In these photographs the two positions of the camera were somewhat less than a meter apart. Note the apparent shift of the ruler with respect to the more distant blackboard.

other and show little or no apparent motion even over thousands of years. For this reason, we call them the fixed stars and we can use them to give us known directions from any point on the earth's orbit.

Unlike the extremely distant stars a near-by star will appear to move relative to the distant ones as we go from one point to another on the earth's orbit. This shift in the apparent direction of the near-by star is just like the shifts you see looking out the car window or the shift you see when you hold out a finger before your eyes and look at it first with one eye open and then with the other. The finger appears to shift its position along a distant wall; and the nearer it is to your face the



3–4. Finding the distance to a near star by geometry. The triangles in the figure are similar, therefore

$$\frac{d}{b} = \frac{l}{s} \quad \text{or} \quad d = b \times \frac{l}{s}.$$

3–5. The shift in position of a near star (Barnard's Star) with respect to the very distant "fixed" stars. Three pictures were made at six-month intervals with the 24-in. telescope at Swarthmore's Sproul Observatory. In this illustration, the pictures have been superimposed. The images of the two very distant fixed stars in the upper right coincide, while the three images of Barnard's Star (lower left) show a horizontal and vertical separation representing two motions of the star. One, indicated by the vertical separation of the three images, is the star's own "proper motion" in a straight line with respect to the fixed stars. The other, the horizontal displacement, is the apparent shift of the star's position as viewed from the two extremes of the earth's orbit. Notice that it goes to the right and then back to the left in successive six-month intervals. It is this displacement that enables us to calculate the distance to the star. The actual shift in the original photo was about 0.03 mm, giving a distance of about 6×10^{13} km.

farther it shifts. Fig. 3–3 is another example of such a shift.*

In Fig. 3–4 we see a simplified version of how an astronomer can fix the distance to a near-by star using the diameter of the earth's orbit as a base line. In order to get the fundamental idea, we assume that the astronomer is lucky: at one moment he finds the near-by star directly lined up with a distant star. The astronomer then waits for half a year so that the earth is now at

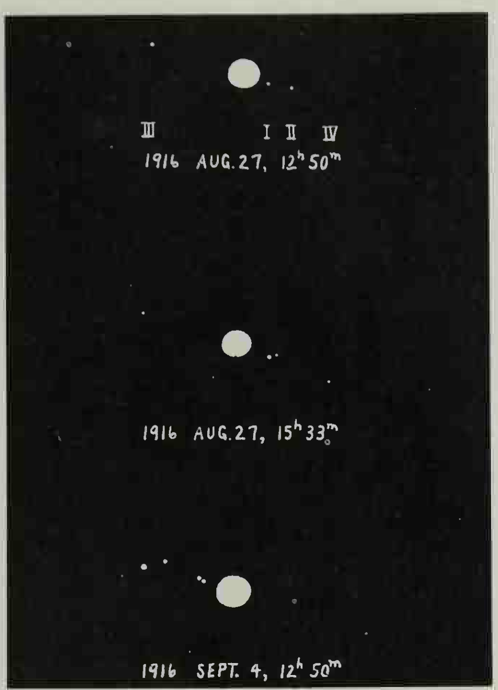
*This apparent shift of one object with respect to another is called parallax. Only when two objects are both extremely far away or both at the same position do they show no parallax. Only then is there no shifting with respect to each other when we move.

the other end of the base line. Then he takes a picture, pointing his camera at the far star so as to get the same direction again. Because the earth has moved, the two stars are no longer in line; and consequently he obtains two separate images on his photographic plate. Because the direction to the distant star is the same, the separation of the images on the photographic plate is related to the length of the earth's orbit in just the same way that the length of the path of light in the camera is related to the distance to the near-by star. You can see this from the similar triangles in the figure. Of course, if no distant star ever lines up with the near-by star (Fig. 3–5), the job is a little bit harder, but the method is essentially the same.

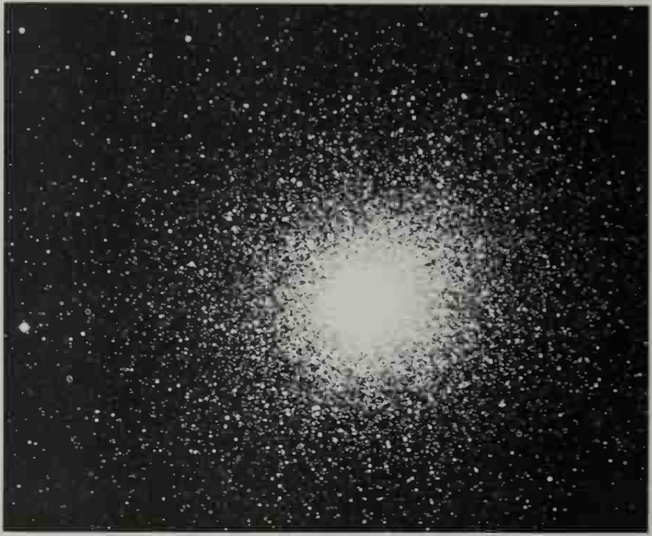
To make things clear in Fig. 3–4 we have used a near "star" whose distance is only a few times the diameter of the earth's orbit. In fact, there are no such stars. Even with the earth's orbit as the base line, the change in direction of the nearest star is very small, for that star is about 135,000 orbital diameters (about 4×10^{16} meters) away. The distances to only a few hundred stars are small enough to be measured in this way. For greater distances in the universe, we must use other methods, one of which is discussed briefly in Section 4–3.

The planets are close enough so that even a small telescope will show most of them as round discs. (Fig. 3–6.) This is clearly what you would expect if they are really globes whose sizes are of the same order of magnitude as our earth. But not even the Palomar telescope shows up any such clear discs for the stars. They are too far away. Astronomers have special means of measuring the sizes of stars. For the present we want to stress that we have a consistent picture. The stars are very far away, and hence they show no parallax, no disc. Closer to us they would be so many glowing suns.

Table 1 on page 28 shows the range of distances from our own size to larger and smaller sizes that can be measured with rulers, geometry, and light. The large distances beyond those that can be measured in this way are shown in Table 2 on page 30. The methods by which the geometrical measurement of distance has been extended to these distances beyond the reach of our best base line and best angle measurements are many and ingenious.



3–6. Jupiter and its moons. As we go to distances beyond the earth it is necessary to use telescopes to extend our senses. The planet Jupiter is visible to the unaided eye, but looks like a bright star. Have you ever noticed in viewing it that it doesn't seem to twinkle as much as a star? The illustration above shows the planet as photographed at three different times with a small telescope. The four satellites that show up are the ones discovered by Galileo and are often referred to as the Galilean moons. (Courtesy: Clyde Fisher and Marian Lockwood, "Astronomy," John Wiley & Sons, Inc.) In the illustration below, we see the planet as photographed through Palomar's 200-in. telescope. Note how much more detail is visible. Even the shadow of one of the moons can be seen. Jupiter has a diameter of the order of 10^8 m and is at a distance of the order of 10^{12} m from the earth.



3–7. The globular star cluster in the constellation Hercules is so far from the earth that its distance cannot be measured by geometric methods. This photograph, made with the 200-in. Palomar telescope, shows one of the finest examples of a globular cluster in the northern sky. It is visible to the naked eye, under good conditions, as a small hazy patch of light. Its diameter subtends an angle of about 18 minutes at the eye. Actually, as you can see in the photograph, it consists of thousands of stars, most of which are larger and more brilliant than the sun. Although they seem to be crowding one another, in reality they are separated from each other by an average distance that is about 50,000 times the distance from the earth to the sun. The great distance between the cluster and the earth makes the stars appear close together. How could we go about measuring how far from the earth these stars are located?

Table 1. Distance

Orders of Magnitude of Lengths Found with Rulers, Geometry, and Light

Length in Meters	Associated Distance	Length in Meters	Associated Distance
10^{18}	Greatest distance measurable by parallax	10^{7}	Air distance from Los Angeles to New York
10^{17}	Distance to nearest star		
10^{16}		10^{6}	Radius of the moon
10^{15}		10^{5}	Length of Lake Erie
10^{14}		10^{4}	Average width of Grand Canyon
10^{13}	Distance of Neptune from the sun	10^{3}	One mile
10^{12}	Distance of Saturn from the sun	10^{2}	Length of football field
10^{11}	Distance of Earth from the sun	10^{1}	Height of shade tree
10^{10}	Distance of Mercury from the sun	10^{0}	One yard
10^{9}	Mean length of Earth's shadow	10^{-1}	Width of your hand
	Radius of the sun	10^{-2}	Diameter of a pencil
10^{8}	Mean distance from Earth to the moon	10^{-3}	Thickness of windowpane
	Diameter of Jupiter (Fig. 3–6)	10^{-4}	Thickness of a piece of paper
10^{7}	Radius of Earth	10^{-5}	Diameter of red blood corpuscle

A collection of a hundred billion distant suns make up what is known as our galaxy. Our own star, the sun, is probably a quite ordinary, though rather elderly, family member. Beyond our galaxy comes a great collection of other galaxies, cousins of our own, dotting the heavens as far as our greatest telescopes can reach. They spread out in all directions, looking fainter and smaller the greater their distance, but they are recognizably similar to our own. The nearest of these is the Great Nebula of Andromeda, which you can just see with the naked eye on a dark, clear night. (See Figs. 3–8 and 3–9.) Nearly a billion distant galaxies are scattered throughout the universe, according to estimates based on photographs of the sky taken with the big Palomar telescope. How many more there may be we are unable to say.

3–8. The Great Nebula in the constellation Andromeda. This enormous island universe of stars, which is similar to our own galaxy, is visible to the unaided eye under favorable conditions as a hazy patch of light, subtending an angle of about 3 degrees. It is the most remote object that is visible to the unaided eye. It is of the same order of size as the Milky Way, about 100,000 light-years in diameter. This photograph, made with the 48-in. Schmidt telescope at Mt. Palomar in California, also shows two satellite galaxies of the Great Nebula (center right and center left).

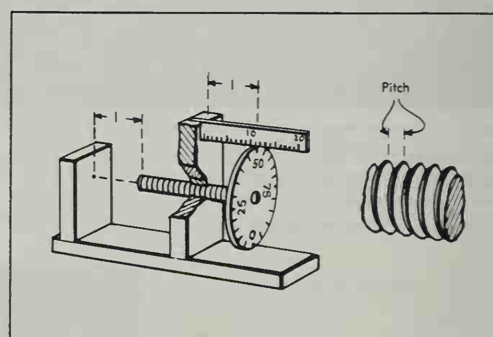
3–9. Cluster of galaxies in the constellation Coma Berenices. This photograph was made with Palomar's 200-in. Hale telescope. If we examine closely the light specks visible in this picture, we note that some of the images have a shape that resembles that of the Andromeda Nebula or one of its satellite galaxies in Fig. 3–8. These are indeed nebulae. They show different shapes and orientations. From the size of these images, this cluster of galaxies can be estimated to be about 20 or 30 times farther away from us than the Great Nebula in Andromeda. To an observer in one of these galaxies, the Andromeda Nebula and our own galaxy, the Milky Way, would appear as two neighboring members of a distant cluster.

Table 2

Orders of Magnitude of Distances Too Large to Measure by Geometric Means

Length in Meters	Associated Distance
10^{25}	Distance to farthest photographed object (a galaxy)
10^{24}	Domain of the galaxies
10^{23}	Domain of the galaxies
10^{22}	Distance to the Great Nebula in Andromeda (nearest galaxy)
10^{21}	Distance to the smaller Magellanic Cloud
10^{20}	Distance of the sun from the center of our galaxy
	Distance to globular star cluster in Hercules (Fig. 3–7)
10^{19}	Distance to the North Star (Polaris)

3–10. A micrometer caliper (above). Very small lengths, such as the thickness of a piece of paper, can be measured with this instrument. The basic part of this device, as shown in the simplified version sketched below, is a screw. Note the scale around the barrel, corresponding to the disc below, which enables us to measure a fraction of a turn. How does a screw help us to measure a small distance accurately?

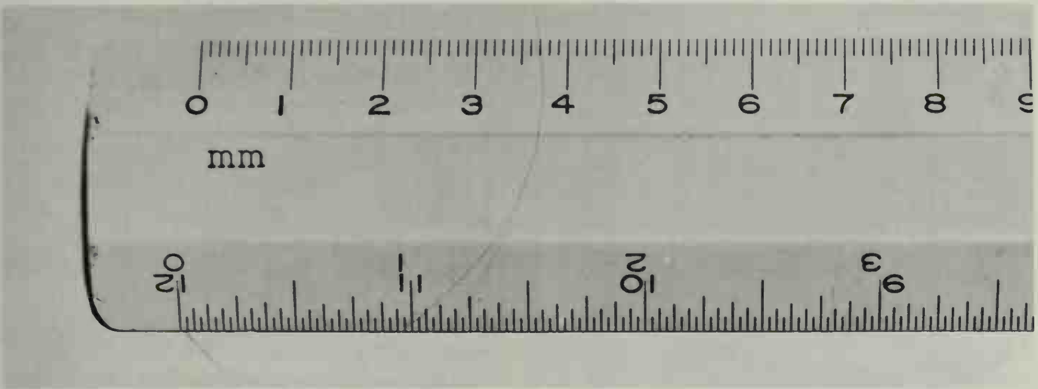


3–3. Small Distances

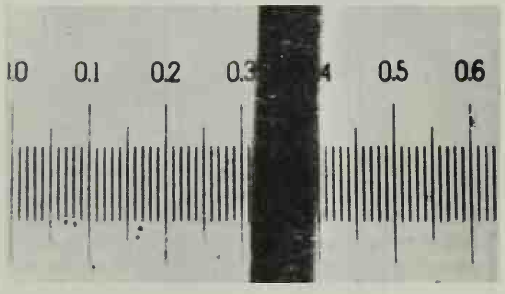
If we go in the other direction toward the very small, we can still use straightforward geometrical methods. It is not hard to measure the thickness of a thin sheet of paper, if you have many of them. Stack up a hundred sheets, use a ruler to measure the stack; then you have marked off on the ruler a hundred times the thickness. This obvious indirect method is similar to what is often done in physics. Of course if the sheets of paper are very different in thickness, the result will refer not to any real sheet, but to an average of the thicknesses present. For many purposes this is good enough. What we obtain is the thickness of a sheet, assuming them all to be alike.

This page-thickness example shows how we can extend the basic idea of counting or spacing off to small distances. Another extension of counting to small distances is found in the use of a screw thread. If a screw is turned through one revolution in a fixed nut, it advances only by the distance between successive threads, the pitch of the screw. By dividing the turn into say a hundred parts, you can divide the advance of the screw into a hundred equal parts as well. This is the basis of the machinist's micrometer (Fig. 3–10). Other similar tricks will help a little, but to go further toward the very small we need to use amplifying devices, of which the most familiar is the microscope. With it, we can see small objects, and measure them by placing tiny "rulers" right beside them (Fig. 3–11). Again, the laying-off and counting method is at work.

The atom and its sub-units are so small that the ordinary microscope is no longer of help, for light itself is not a delicate enough probe. Newer sorts of instruments, and again a set of still more indirect but convincing methods, carry us down to the smallest distances about which we have any real knowledge. Some of the methods are simple enough for you to carry out. See your laboratory guide. Table 3 gives some idea (in terms of orders of magnitude) of the amount by which we can extend our everyday notions of distance and size into the very small.



3–11 (a). The size of a human hair. In this photograph a hair has been placed across a millimeter scale. It is barely visible. How many hairs would have to be placed side by side to fill the space between two adjacent millimeter marks on the scale?



3–11 (b). Here the hair has been photographed on a very small ruler with the aid of a microscope. Each of the smallest divisions on the ruler is one hundredth of a millimeter. The microscope has made it possible for us to measure the diameter of the hair more accurately. How accurate was your estimate from Fig. 3–11 (a) of the number of hairs that would be required to cover a millimeter?

Table 3

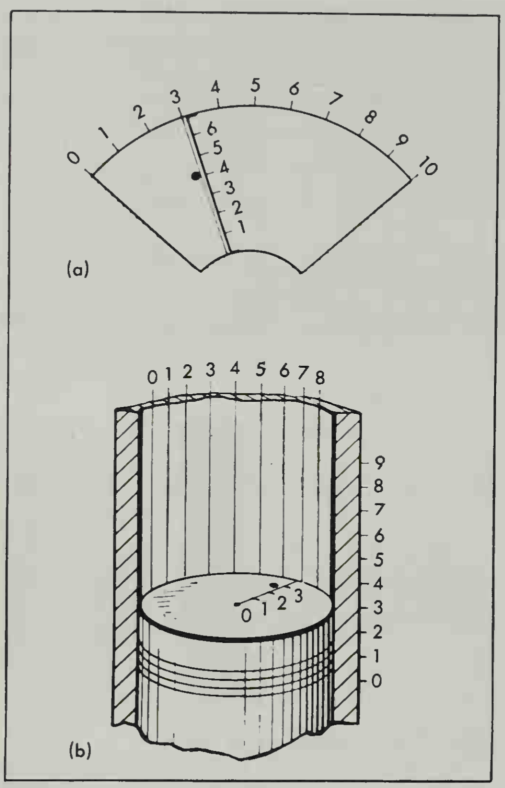
Orders of Magnitude of Distances Too Small to Measure by Geometric Means

Length in Meters	Associated Distance
10^{-6}	Average distance between successive collisions (mean free path) of molecules in the air of a room
10^{-7}	Thickness of thinnest soap bubble still showing colors
10^{-8}	Average distance between molecules of air in a room
10^{-9}	Size of molecule of oil
10^{-10}	Average distance between atoms of a crystalline solid
10^{-11}	
10^{-12}	Average distance between atoms packed in center of densest stars
10^{-13}	
10^{-14}	Size of largest atomic nucleus
10^{-15}	Diameter of proton

3–4. The Dimensions of Space

The fact that space has three dimensions is usually demonstrated by pointing out that three separate measurements are needed to locate an object in space. In the room in which you are now sitting, for example, you can locate any point by specifying its distance from one wall, its distance from a second adjacent wall, and its height from the floor. We can say this with complete confidence, even though we may not know the shape of the room. We assume that it has at least two straight walls that meet in a corner of some kind. If walls and floor meet at right angles, they represent what are called rectangular or Cartesian coordinates. If the corners are all right-angled, the calculations may be simpler, but they are no better or no worse than any other kind. If the room is circular, three numbers will still do the job, although the calculation is different.

In any case, three numbers — and the rules that say what they mean — define any point, and only that point. This is merely one way of expressing the fact that space is three-dimensional. It is, however, not always the most interesting or the most informative way.



3–12. (a) Locating a point on a surface. (b) Locating a point in a volume.

We may approach the three dimensions of space in another manner. If you take a point — the point of your pencil is ideal — and move it, you create a line. Any position on the line can then be specified by stating its distance from the beginning of the line. The line, in other words, has one dimension.

If you now take the line, and move it, you create a surface. A windshield wiper is an excellent example of this: the line of its rubber edge, on each sweep, marks out a surface on the windshield of the car. To locate a point on this surface, you need two numbers, one to give the position of the wiper when it lies across the point, and the other to state how far out along the wiper the point is. The surface, in other words, is two-dimensional. [See Fig. 3–12 (a).]

In the engine of your car there are cylinders and pistons. The piston head is a surface. As it moves up and down from one position to another inside the cylinder, it sweeps out a volume. To find a point in this volume, we need three figures — two to define a point on the piston head, and a third to tell how far the piston is from one end of its stroke. [See Fig. 3–12 (b).]

Having now seen how a moving point generates a line, how a moving line generates a surface, how a moving surface generates a volume, what happens if we take the next step? What if we move a volume? The result is not what we might expect. A volume only sweeps out another volume, no different in kind from the volume swept out by a surface. We have run out of dimensions. Space, it seems, offers us only three upon which to work. Space is apparently three-dimensional, and no more.

There is still another way of looking at dimensionality. In this view, the pertinent characteristic of a line is that we can move along it from point to point without interruption — without lifting our pencil, as it were. But if one point is removed, we can no longer move directly from a point on the line to any other point beyond the gap. In effect, the line now is cut.

Removing a point from a surface, such as the floor of the room, does not hinder us. We can move from any point on the floor to any other point merely by going around the missing point. But cut the floor along a line so that it now has two disconnected areas. If we are on one side of the cut, we cannot go directly to the points on the other side of the boundary.

Finally, within the room as a whole, a full surface — a wall — is needed to prevent crossing from one point to another. But here again we come to the end. Any closed volume can be walled into two separate volumes, but we can go no further (unless the mathematicians invent new spaces, and they often do exactly that).

What we have just done can be stated in simple form: a point (with zero dimension) cuts a line; a line (with one dimension) cuts a surface; a surface (with two dimensions) cuts a volume or space. A volume (with three dimensions) merely cuts another volume.

All this may appear unimportant, or at best not important enough to warrant three repetitions. As we go along, however, we shall find reasons to use each of these aspects of dimensionality. We will be working in physics with things of no dimensions, of one dimension, and of two dimensions, as well as with physical space and its three dimensions.

Time, for example, has one dimension. It is specified by one number. We say, "Ten minutes from now." It is measured out by the passage of zero-dimensional instants. And in passing through time, we must pass through all the instants, one after another. For example, there is no way of getting from 8:30 a.m. to 8:32 a.m. without passing through 8:31 a.m. Each of these facts is significant; each is a characteristic of something having only one dimension.

3–5. Measuring Surfaces and Volumes

The clue for the measurement of surfaces and volumes lies in the way we measured distance. Lay off a convenient unit of area and simply count how many times the unit fits into the surface to be measured. By subdividing sufficiently, it is possible to fit the unit, or its subdivisions, into all the corners and curves of any surface with as much accuracy as you wish. (Fig. 3–13.) The process is similar to laying a tile floor.

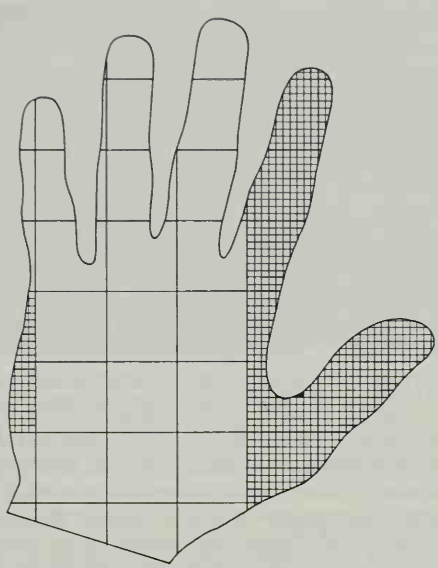
The convenient unit always used for surfaces is a square whose edge is a standard unit of length. Since we use meters for length, we have as a unit of surface the square meter (m^2).

We can measure volume in the same way, fitting little cubes into every portion of the volume to be measured, until it is filled up. Here the unit we shall use is the cubic meter (m^3). Familiar divisions of these basic units are the square centimeter (cm^2) for area and the cubic centimeter (cm^3) for volume. How many square centimeters are there in 3 m^2? We know

$$1 \text{ m} = 100 \text{ cm};$$
$$\text{so} \quad 3 \text{ m}^2 = 3 \times 100 \text{ cm} \times 100 \text{ cm} = 3 \times 10^4 \text{ cm}^2.$$

Of course, the fitting of these little squares or cubes into irregular surfaces or volumes is not the only way to measure area or volume. Standard containers of convenient shape are usually on hand, and an odd-shaped volume, such as a milk bottle, might be measured by filling it with water and then pouring the water into a standard container or two, eventually using subdivisions made in some geometrical way on a container of simple shape like the familiar graduated cylinder.

The area of an irregular surface can be found by weighing a paper pattern cut to fit the surface neatly. One then compares the weight of the cut



3–13. Measurement of a surface. In measuring the area of an irregular surface, such as the hand pictured above, we use the same method that we used in measuring a distance. First we lay off our units on the surface. To measure areas smaller than our unit area, we subdivide our unit. In the illustration, the unit is being subdivided to measure small irregularities. As you can see, there are additional small areas that will not fit these subdivisions. In such cases, we can subdivide the units as many times as we wish until we reach a point beyond which it is useless to go because the subdivisions become too small to see.

pattern with the weight of a measured square of the same material to find the area.

3–6. On the Limitations of Measuring

We have founded all our measurements on one simple scheme. To measure the size of some physical quantity, length, or time, you first choose a unit — any length or time you wish will do. Then to measure an interval larger than the unit, just "lay off" the unit as many times as it will go into the required interval. This is what we naturally do with a ruler. For anything left over after the count, or for any amount we want to measure which is smaller than the unit, we simply divide the unit into smaller equal parts, sub-units you may call them, and take as many of them as we need to match the given magnitude. We measure a box and find its length to be 20 cm and something left over. Dividing our centimeter unit into tenths, we find that the part left over contains three of these sub-units; so we say the box is

20.3 cm long. It is not hard to see that this method will work for any length that we want to measure. For we can make the divisions finer and finer until irregularities in the edge of the box we measure, or in the markings on our ruler, limit the fineness of our measurements of its length.

Some measurements are not subject to the process of making smaller and smaller subdivisions for greater and greater accuracy. The counting of the number of people in a room, for example, has a natural unit, the individual. Here the whole question of smaller and smaller subdivisions is irrelevant. Unlike time and space, matter does have known natural units. This is the real essence of modern physics. The natural units of matter are its building blocks, atoms and their few parts, which combine in so many ways to make up the whole of the material world — stars and sea, pencil and paper, skin and bone. We do not know whether space and time do or do not have such natural units. We only know that we have not run into them. Until we find such units (if we ever do) we will freely use any subdivision of our arbitrary unit of measurement to represent time and space.

We have just looked at the problems involved in the basic method of measurement by counting. In many real measurements a second type of problem arises. A measurement that is made by an indirect method is always based on special assumptions. In measuring the thickness of a piece of paper, for example, we made an assumption that the paper was uniform. The measurement of large distances by triangulation also involves an assumption — one we are pretty familiar with in everyday life. We assume that the line of sight — which is the line that light travels to get to the eye from the object — is a straight line. Only if this is right will our method of sight-triangulation work. Commonly we check the straightness of a board by sighting along it. We seem to accept the straightness of the path of light. Of course, it can deceive us and often does. The heat-shimmer you see above a hot radiator or a sun-warmed surface tells you that here are sight-paths which are not straight and are constantly changing. If we wish a reliable answer when measuring long distances by triangulation, we must avoid looking through heated, disturbed air. We cannot measure the distance to a star by this means on a night when the star is twinkling very much as a result of changing air currents from the warm surface of the earth. We want a clear, still night, with the star well up in the sky.

Another assumption involved in measuring by triangulation is that the laws of geometry are correct. They cannot be taken for granted, however. All assumptions that we make in measuring must be tested. The results of geometry and the straight path of lines of sight have been well tested, largely by the success of the whole picture we can build up. But we must always be on the watch, especially when using indirect methods in measuring things far from everyday experience, to see if such traditional assumptions can still be relied upon.

We noted earlier that we must do our best to understand the limitations of our instruments, including our senses. The problem of measuring the sizes of planets and stars illustrates this point. When we look through a telescope at various planets, they have various sizes; they appear as discs of various diameters. Stars seen in telescopes also appear to have some diameter, but the diameter does not change as we look from star to star. Instead it depends on which telescope we use and on which way we point it. The apparent size of stars seen in telescopes arises from the behavior of telescopes, not from the real size of stars. (See Fig. 3–14.) We have run into a limitation of our instrument, one which we can later understand, and we must get the information about the size of stars some other way. All indirect methods of measurement have limitations, and no one method works for all cases.

Even the method of using standard lengths has its problems. In very precise land surveys, for example, the temperature of the steel tapes used is measured meter by meter in order to correct for expansion or contraction. Here, because we employ physical objects, the direct method must be carefully scrutinized.

3–7. Significant Figures

Numbers and their combinations by means of arithmetic give us an exact way of speaking about quantity. In physics, however, there are limits to our accuracy of measurement, and they in turn place limits on our use of numbers to record our measurements.

We have learned that the use of a large string of zeros on either side of the decimal point, to express the order of magnitude of a quantity, is unnecessary. Every quantity can be written as a decimal number between one and ten multiplied by the appropriate power of ten. Instead of writing the radius of the earth as about 6,370,000 meters, therefore, we write it as 6.37×10^6 meters. Likewise, the diameter of a hair is about 0.00003 meter, which we write as 3×10^{-5} meter.

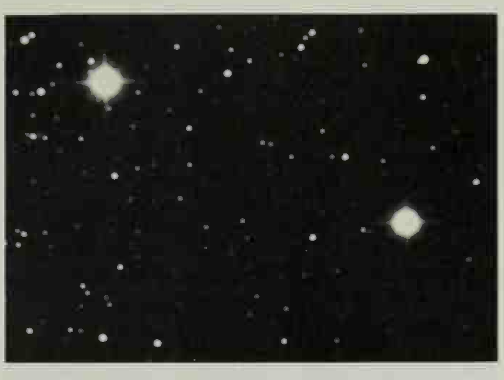
Now, in this way of writing numbers, we show the limited accuracy of our knowledge by omitting all digits about which we have no information. Thus, for the earth's radius, when we write 6.37×10^6 m and not 6.374×10^6 m or 6.370×10^6 m, we are saying that we are reasonably sure of the third digit but have no idea of the value of the fourth. The number of digits about which we do feel reasonably sure is called the number of *significant figures*. In the example of the hair, we have indicated only one significant figure. This means that we think three is a reasonable value, but we are not at all sure of the next digit (second significant figure).

A physicist who makes a measurement must estimate its reliability, and the simplest way of expressing that reliability is by writing the proper number of significant figures. To write additional figures that have no meaning is worse than a waste of time. It may mislead people who use those figures into believing them.

It is clear that the greater the accuracy of our measurements, the larger the number of significant figures we can use. When we write four significant figures, we imply that a fifth digit would have no meaning. If our accuracy were ten times greater, we would use another significant figure. The most careful physical measurements, using the highest available accuracy of the primary standards, still fall short of having twelve significant figures.

Because the numbers used in physics reflect the limitations of measurement, we modify our ideas of arithmetic slightly so as to make sure that we do not write meaningless digits in our answers. Suppose we make the following time measurements — 27.8 hr, 1.324 hr, and 0.66 hr — and we want to find their sum. Paying no attention to significant figures, we might write

$$\begin{array}{r} 27.8\ \text{hr} \\ 1.324\ \text{hr} \\ \underline{0.66}\ \text{hr} \\ 29.784\ \text{hr} \end{array}$$



3–14. This illustration, an enlargement of a small section of Fig. 3–7, shows two star images in detail. The images of the two brightest stars display four rays, while the smaller stars appear as small, irregular shapes. The four neat rays on the bright stars are due to the out-of-focus image of a four-armed support within the telescope. Even the roundish shapes of the images of the fainter stars do not depend on the stars, but on the nature of the telescope, the atmosphere, the photographic plate, and the light. These cause the light from a distant star to blur rather than focus to a sharp point. The true star image in each case would be considerably smaller than the blurry spot. For this reason, the star images of Fig. 3–7 appear much more crowded than the stars which make them. In the same way, the stars that form our Milky Way cannot be separated by the unaided eye.

What is the meaning of this result? In any number obtained by measurement all the digits following the last significant one are unknown — for example, the hundredths and thousandths place in the first measurement above. These unknown digits are not zero. Clearly if you add an unknown quantity to a known quantity you get an unknown answer. Consequently, the last two digits in the sum above are in fact unknown. In this case, then, we should round off all of our measurements to the nearest tenth so that all the digits in our answer will be significant. This gives

$$\begin{array}{r} 27.8\ \text{hr} \\ 1.3\ \text{hr} \\ \underline{0.7}\ \text{hr} \\ 29.8\ \text{hr} \end{array}$$

Since the first measurement is known only to the nearest tenth of an hour, we know the sum only to the nearest tenth of an hour.

Subtraction of measured quantities works the same way. It makes no sense to subtract known and unknown quantities. Particular care must be taken in subtracting two numbers of nearly equal magnitude. For example, suppose you wish to find the difference in length of two pieces of wire. One you have measured to be 1.55 meters long and the other 1.57 meters long.

$$1.57 \text{ m} - 1.55 \text{ m} = 0.02 \text{ m} = 2 \times 10^{-2} \text{ m}.$$

Notice that we do not write the answer as 2.00×10^{-2} m, since we are somewhat uncertain about each of the last digits in the original measurements. The difference certainly has only one significant figure, and we would not be too surprised if the difference were either twice as large or zero instead of 2 cm. Subtraction of nearly equal quantities destroys accuracy. For this reason, you sometimes need measurements much more accurate than the answers you want. To avoid the difficulty of making more accurate measurements, we would put the two wires side by side if possible and measure the difference directly with a micrometer screw rather than use the difference between two large numbers.

Now what about multiplication? How do we modify it to take account of the limitations of measurement? Suppose we wish to find the area of a long strip of tin. With a meter stick we measure its width to be 1.15 cm and its length to be 2.002 m. Here we have three-significant-figure accuracy in our width measurement and four-significant-figure accuracy in our length measurement. To get the area we multiply length by width. Paying no attention to significant figures, we get

$$\begin{aligned} A &= 2.002 \text{ m} \times 1.15 \times 10^{-2} \text{ m} \\ &= 2.30230 \times 10^{-2} \text{ m}^2. \end{aligned}$$

But now think of the meaning of this answer. When we measured the width we wrote 1.15 cm because we were not sure that the real width might not be a bit bigger or a bit smaller by perhaps 0.01 cm. If in fact the width is that much bigger, we have made a mistake in the area by the product of this extra width times the length, that is,

$$\begin{aligned} \text{Error} &= 0.01 \times 10^{-2} \text{ m} \times 2.002 \text{ m} \\ &= 0.02 \times 10^{-2} \text{ m}^2. \end{aligned}$$

Thus we see that we have an uncertain number in the hundredths place, which means that our original evaluation of the area may already be in error in the third significant figure. All the figures we write beyond the third have no significance. The proper way to express the answer is 2.30×10^{-2} m^2, for when two numbers are multiplied together, their product cannot have more accuracy (or more significant figures) than the less accurate of the two factors. Don't think that your results are improved by carrying out simple arithmetical operations to many figures.

What has been said about multiplication applies equally well to division. Never carry a division out beyond the number of significant figures in the least accurate measurement you are using.

It should be noted that numbers that are not the result of measurement may have unlimited accuracy and may be taken to any degree of accuracy required by the nature of the problem. For example, if an area was measured and found to be 3.76 m^2, twice that area would be $2 \times 3.76 \text{ m}^2 = 7.52 \text{ m}^2$.

We have seen how to handle numbers when they represent physical quantities. But we have by no means told the whole story of accuracy in measurement. The use of significant figures sometimes raises difficulties that would lead us into a detailed study of the theory of errors. However, the idea of significant figures helps us avoid misleading numbers and unnecessary calculation.

Every physical quantity must have: a unit, to tell what was counted; an order of magnitude; and a statement about its reliability, which for the present we can make in a rough way by writing only the correct number of significant figures. There is no technique in physics more important than the writing of physical quantities with all these facts made clear.

FURTHER READING

Gamow, George, *One, Two, Three . . . Infinity.* Mentor Books, 1957 (pp. 257–261).

Helson, W. H. I., and Kilpatrick, F. D., "Experiments in Perception." *Scientific American,* August, 1951 (p. 50). Some remarkable optical illusions illustrating the fallibility of our senses.

Lee, Oliver J., *Measuring Our Universe.* Ronald Press, 1950. How physicists and astronomers measure distances within the atom and in outer space.

Moore, Patrick, *The Story of Man and the Stars.* Norton, 1954.

Whipple, Fred, *Earth, Moon and Planets.* Harvard University Press.