MOTION ALONG A PATH

CHAPTER 5

A FREIGHT TRAIN is rolling down the track at 40 miles per hour. Around the bend a mile behind, a fast express appears, going at 70 miles per hour on the same track. The express engineer slams on his brakes. With the brakes set he needs two miles to stop. Will there be a crash? What we are called upon to do here is to predict where the two trains will be at subsequent times, and to find in particular whether they are ever at the same place at the same time. In a more general sense, we are asking about the connections between speeds, positions, and times.

The general subject of such relationships is called *kinematics.* In studying kinematics we do not concern ourselves with questions such as "Why does the express train need two miles to stop?" To answer such a question we would need to study in detail how the brakes slow down the train. Such questions as these will be considered in Part III on Mechanics. Here, we just consider the description of motion. We shall start with the discussion of motion along a given path without considering the position and direction of the path in space. Then in the next chapter we shall extend the discussion to describe the path.

In both of these chapters we shall draw on our ability to measure time and distance, for all motion is the changing of distance as time goes on. Usually we shall not think consciously of the time and distance measurements, but without them we would in fact be talking words without meaning.

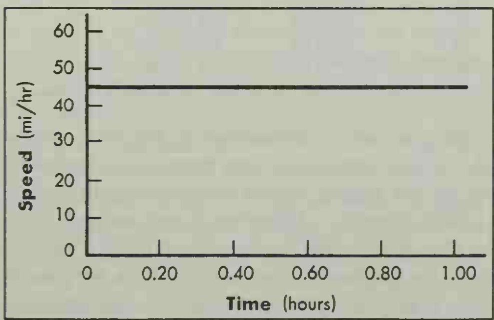
5–1. Speed and Distance

For a body moving with a constant speed, the relationship of time, speed, and distance is expressed simply. If we let d stand for the length of the trip, v for the speed, and t for the time needed for the trip, the equation

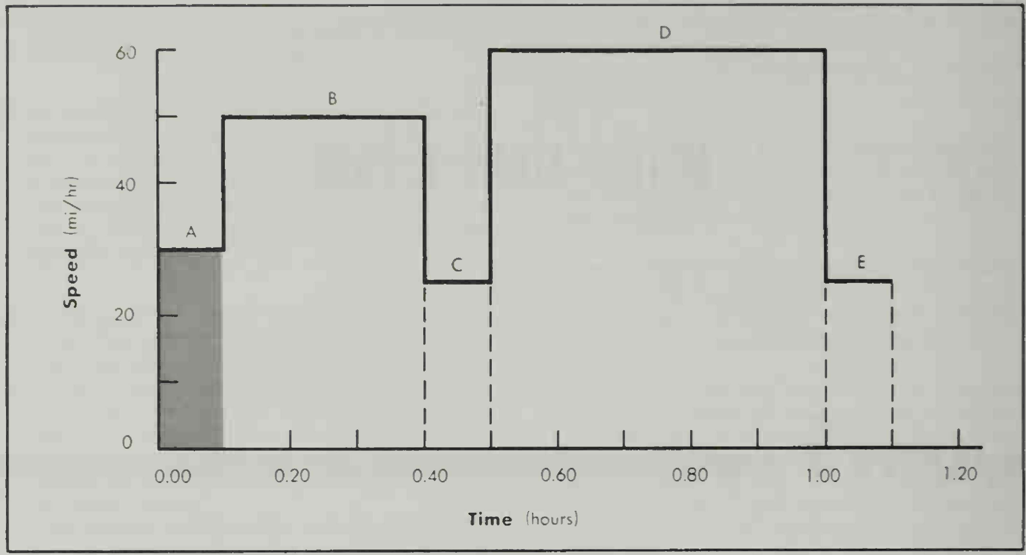
$$d = vt$$

relates these quantities for all cases of constant speed.

It is often convenient to use a graph to represent motions. Fig. 5–1 shows a graph of the speed versus time for a car which travels at 45 mi/hr. Taking some horizontal position on the graph, such as that corresponding to 0.20 hr, we find a reading on the vertical (or speed) axis of 45 mi/hr. In fact, we find 45 mi/hr for *any* time we select.



5–1. The speed of a car moving steadily may be graphed as a horizontal line.



5–2. The motion of a car moving at different speeds during different time intervals. The distance covered in any interval is measured by the area enclosed.

A more complicated motion is described in Table 1. To compute the distance traveled during the first time interval (0.10 hr long) we use the equation $d = vt$. The result is 3 miles. We can perform a similar calculation for each succeeding interval, and add the results to find that the total length of the trip is 53 miles.

Table 1

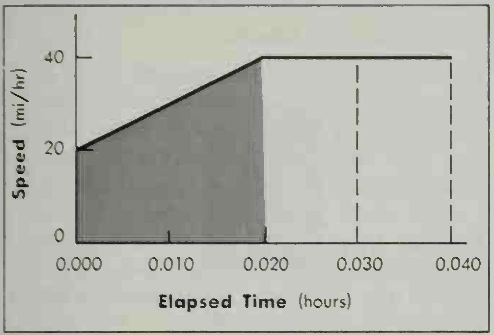
Motion of a Car at Variable Speed

Time interval number	Duration of interval	Speed during interval	Distance in miles
A	0.10 hr	30 mi/hr	3
B	0.30 hr	50 mi/hr	15
C	0.10 hr	25 mi/hr	2.5
D	0.50 hr	60 mi/hr	30
E	0.10 hr	25 mi/hr	2.5

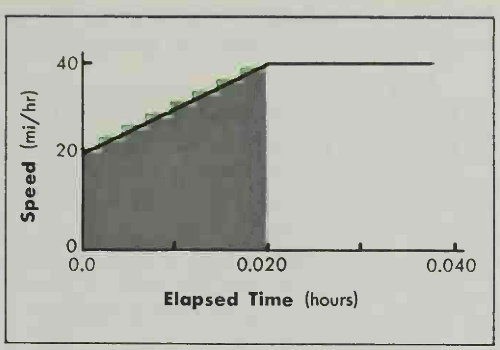
This motion is represented in Fig. 5–2. Actually, a real object could not move *exactly* according to this graph. Speed cannot increase in such sudden "jumps." However, a real car can make its changes of speed relatively rapidly. In that case, the graph of its motion will look very much like Fig. 5–2. We shall ignore the impossibility of sudden jumps in this discussion, so that we can keep our graph simple.

One great convenience of graphical presentation is that it enables us to see quickly when the car is going fast and when it is going slowly. Thus, the higher speeds occur "high" on the graph of speed versus time. Can the graph also tell us how far the car goes in each interval? The answer is "yes." Let us see how. During any one of the five intervals the car travels a distance given by the equation $d = vt$. In any interval, the height of the graph tells us the speed during the interval, and the horizontal length gives us the time. Thus v times t is the height times the base, or the "area" of the rectangle. This area is shaded for the first interval in Fig. 5–2. The units of these "areas" are different from the more common cm^2 or in^2 because one side of the rectangle is measured in hours, and the other is measured in mi/hr. The product in this case has units of hours × mi/hr = miles traveled.

The vertical axis of the graph represents the speed in mi/hr. But taking a ruler and actually measuring the vertical length in Fig. 5–2 to be 3.0 cm tells us nothing until we know that, for the particular scale of this graph, 3.0 cm represents 30 mi/hr. It is helpful to remember that the graph is a sort of scale drawing. Unlike a map which simply "scales down" distances, the graph



5–3. The speed-time graph for a car which is changing speed during part of its trip. Does the shaded area give the distance traveled during the time interval from 0.000 to 0.020 hours?



5–4. In this figure an imaginary car is alternately moving a little faster and then a little slower than the car of Fig. 5–3 so that, eventually, it covers the same distance.

has different scales in the horizontal and vertical directions — scales which may differ not only numerically, but also in the nature of the physical quantities that they represent and therefore in their units. When we talk about the "height" being 30 mi/hr, we are using the graph in a way that gives the same answer no matter what scale we use in the actual drawing. For example, it makes no difference whether we use 0.5 cm or 1.0 cm to represent 10 mi/hr, but we must know which, and stick to it on any one graph.

Since heights and horizontal distances that we plot on the speed and the time scales of our graph are proportional to the actual speeds and times involved, any two areas on the graph are exactly proportional to actual distances the car moves. This fact often allows us to decide at a glance in which time interval the greatest distance is covered. For example, we can see that the area of the rectangle marked D in Fig. 5–2 is greater than that of any of the other rectangles. Therefore, we know without calculations that the car travels farther in the interval D than in any of the other intervals.

The total distance the car travels in 1.10 hr is obtained by adding up the "areas" of all the intervals in Fig. 5–2.

5–2. Varying Speeds

For the case we have considered, the graph did not give us any really new information because we had a method of computing distances without the aid of a graph. Now we shall use our graphical ideas to help analyze a more difficult problem.

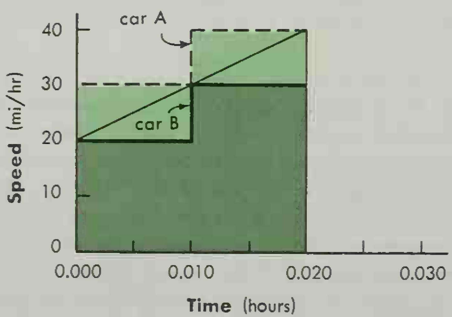
Fig. 5–3 gives a graph of the speed of a car versus elapsed time. Can we tell how far the car goes in the first 0.020 hr? We can try to multiply the speed by the time, but we get into trouble, for we must now choose from a whole range of speeds. On the other hand, using the area under the graph, which worked as an alternate method for motion at constant speed, might also serve here, allowing us to solve graphically a problem that presents difficulties when tackled algebraically. Using the area to find our distance looks reasonable because we can approximate the sloping graph of Fig. 5–3 closely by the one in Fig. 5–4.

The graph of Fig. 5–4 represents the motion of an imaginary car that changes speed in steps (keeping constant speed during each step). Each step brings it to a speed a little greater than the speed of the real car at that instant. Then, while the imaginary car's speed remains constant, the speed of the real car gains on it and passes it. Next the imaginary car's speed increases by another step. The distance covered by the imaginary car is given by the area shaded under the stepped graph of Fig. 5–4. If we make the steps smaller, and more frequent, the two cars would never differ much in speed. Then the shaded area which gives the distance covered by the imaginary car would practically give the distance covered by the real car. And that shaded area, for many steps, is practically the shaded area under the graph of Fig. 5–3 for the real car. If you want to see another discussion which leads to a rigorous proof that the area under this speed-time graph gives the distance traveled, read the material in the box on the next page.

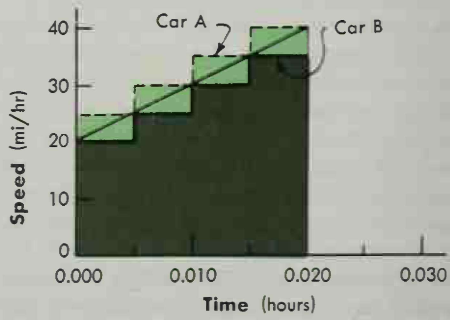
Distance as the Area under the Speed-Time Graph

Here is a more rigorous argument to prove that the distance moved by the real car is the area under the sloping graph in Fig. 5–3. We can bracket the distance covered by the real car between two limits by having two imaginary cars that change their speeds in steps, one of them A always traveling faster than the real car and the other B always slower. Then in a given time A must travel a greater distance than the real car and B must travel a shorter distance than the real car. The distance traveled by the real car lies between those traveled by A and B. First imagine A and B each changing speed in large steps as in Fig. 5–5. B starts with the real car's initial speed. For the first 0.010-hour period, it travels at 20 mi/hr while the real car speeds up from 20 to 30 mi/hr, and for the next 0.010 hour B travels at 30 mi/hr. Meanwhile, A runs at 30 mi/hr for the first 0.010-hour period and then at 40 mi/hr. In a total time of 0.020 hr, B travels 0.20 mi + 0.30 mi or 0.50 mi; while A travels 0.30 mi + 0.40 mi, or 0.70 mi. The distance traveled by the real car must lie somewhere between these two values since it never moves faster than A or slower than B. Thus we have bracketed the real car's travel between the limits 0.50 and 0.70 mi.

Now make the steps smaller and more frequent, as in Fig. 5–6. If you calculate the distances traveled by B and A in the first 0.20 hr (the area under the graphs) you will find they are 0.55 mi and 0.65 mi. This gives a smaller interval between the upper and lower limits than we obtained before. We can continue to decrease the interval between the upper and lower limits for the distance by making the cars change speed in shorter and shorter time intervals. The area representing the upper limit of the distance and the area representing the lower limit become more and more nearly the same, and both of these areas get nearer and nearer to the area under the sloping curve. Mentally continuing this process, we prove that the "area" under the speed-time graph does give the distance traveled by the original car.



5–5. We can "bracket" the distance covered by the car in Fig. 5–3 by imagining two other cars, A and B, that travel with different speeds as shown in this figure.



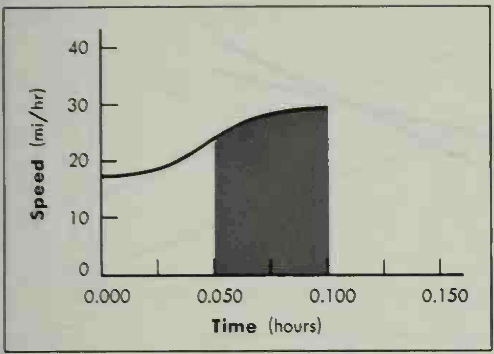
5–6. If the two cars of Fig. 5–5 change their speeds more frequently, it is clear that they approximate the motion of the real car more closely.

Since the "area" under the slant line of Fig. 5–3 is that of a trapezoid with a base of 0.020 hr, and the two heights, 20 mi/hr on the left and 40 mi/hr on the right, we can now say how far the original car goes. The area of the trapezoid gives the distance

$$d = \left(\frac{20\text{ mi/hr} + 40\text{ mi/hr}}{2}\right) \times 0.020\text{ hr} = 0.60\text{ mi}.$$

(You get the same answer by breaking the trapezoid up into a rectangle and a triangle and adding their areas.)

In general, even for more complicated speed-time graphs, such as that of Fig. 5–7, the distance is still given by the "area." For example, for the time interval 0.5 hr to 1.0 hr the distance is given by the shaded area of Fig. 5–7. Even if we are not able to compute the "area" from a



5–7. In general, the distance covered is given by the area under the speed-time graph, no matter how the speed changes.

formula, we can arrive at an approximate answer in other ways. For instance, we can divide the area into small squares and then multiply the area of each square by the number of squares counted, as suggested in Section 3–5.

5–3. Graphs of Distance versus Time

When we drive at a steady speed, the distance we go is proportional to the time we travel: $d = vt$. In other words, since v is constant, the area under the v versus t curve varies with t. At the speed of 60 mi/hr, for instance, in 0.10 hr we go 6 miles; in 0.20 hr, 12 miles, and so on. We can present this information in a table such as Table 2. Or we can use a new graph, the graph of distance d versus time t. The graph will make it easy to find the distance even at a time that is not included in the table.

Fig. 5–8 is the d versus t graph for a speed of 60 mi/hr. Like all direct proportions (see Section 4–1), it is a straight line.

What makes this line correspond to 60 mi/hr is its steepness. How steeply the line rises depends on the speed, which is the proportionality factor between d and t. For example, if the speed were greater, say 80 mi/hr, the straight line would be steeper. It would rise the same distance in a shorter time.

If your car breaks down and you phone the garage, you will probably tell the repairman the location of your car by telling him its distance and direction along the road from some landmark. "It is 5 miles from the blinking light, going toward California," you might say. From now on we shall measure d in a definite direction from some place on which we agree. In this way we can use d to specify position. Motion will still be described by the changes in d as time goes on.

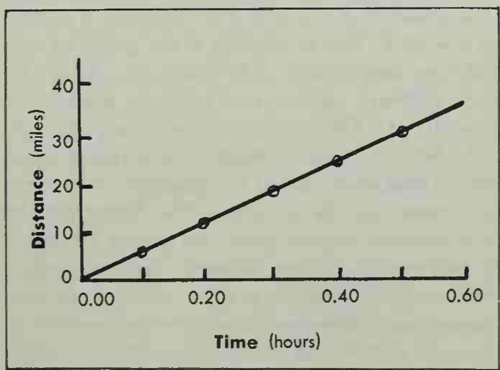
When you drive on a superhighway, you may see a post every mile of the road. For example, the Pennsylvania Turnpike has mileposts which are numbered consecutively 1, 2, 3, etc., starting at the Ohio state line. When we pass a post labeled 176, we know we are 176 miles from the Ohio state line, measured along the road. If at that moment another car is opposite the post labeled 186, it is 10 miles along the road from us, in the direction away from Ohio.

We shall use this idea to help us make a graph which shows the positions of *two* cars. We describe the position of either car by giving its distance d along the road from some reference point like the state line or the place where a trip begins. We can then make a graph of d for each car at various times, as is done in Fig. 5–9 for one example.

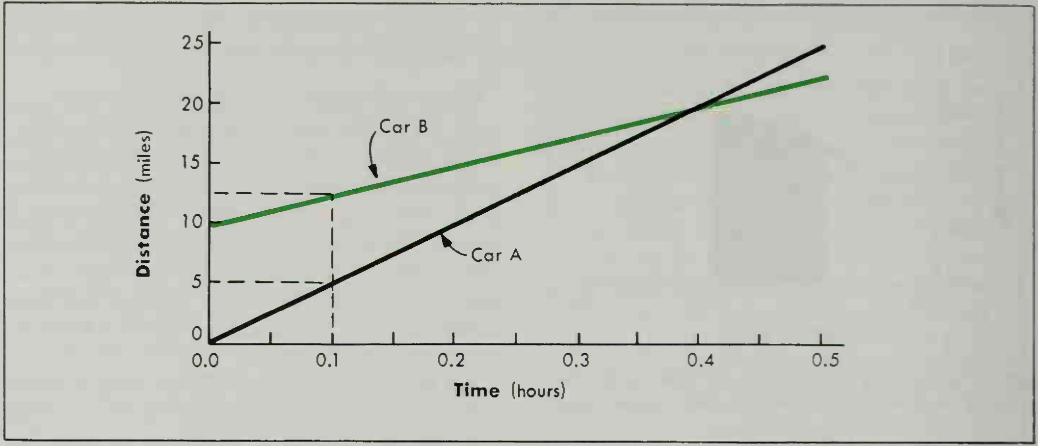
Table 2

Distance-Time Relations for Steady Speed

Elapsed time	Distance covered
0.10 hr	6 mi
0.20 hr	12 mi
0.30 hr	18 mi
0.40 hr	24 mi
0.50 hr	30 mi



5–8. The distance-time graph for a steady speed is a straight line.

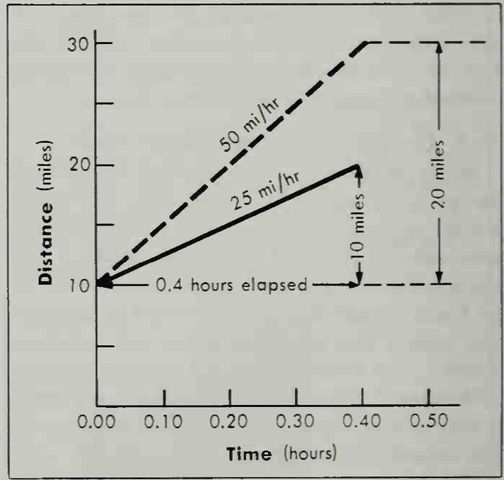


5–9. At what time will car A overtake car B?

From the graph we can calculate the speeds of the cars. In 0.1 hr, for example, car A goes from the position $d = 0$ to $d = 5$. It moves 5 mi and its speed is therefore 50 mi/hr. In the next 0.1 hr, it again goes 5 mi from $d = 5$ to $d = 10$; its speed is still 50 mi/hr. Because the graph is a straight line, the distance car A moves is the same for every 0.1 hr; therefore the speed of A is 50 mi/hr all the time. Car B also has constant speed. In each 0.1 hr it moves 2.5 mi, from $d = 10$ to $d = 12.5$ in the first 0.1 hr, from 12.5 to 15 mi in the second, and so on. Its speed is therefore 25 mi/hr.

In addition to the speeds, the graph tells us more. It says that car B starts 10 mi ahead of A, but A catches up. After 0.1 hr A is at $d = 5$ and B is at $d = 12.5$. A is therefore only 7.5 mi behind B. By the time 0.5 hr has passed we see that A is ahead of B. It is at $d = 25$, while B is only at $d = 22.5$. Just by looking at the graph we can tell how long it took A to catch up. At 0.4 hr both cars are at the same position, actually at $d = 20$; at 0.4 hr, therefore, A was just passing B.

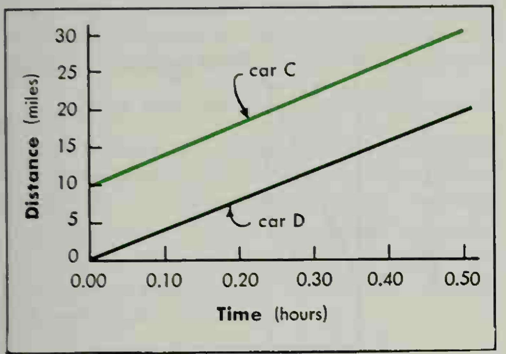
In Section 5–1 we saw that on a graph of speed versus time we could tell at a glance at what times the speed was the greatest. The higher up the line occurred on the graph the greater the speed it represented. Now, however, we are dealing with a quite different graph — that of *distance* versus time. The speed is involved only indirectly in such graphs and is *not* shown by the height of the line above the time axis. For example, in Fig. 5–9 the line for car B is above that for car A in the entire interval from 0.00 hr to 0.40 hr, although car B is being overtaken during this interval and is certainly the slower of the two.



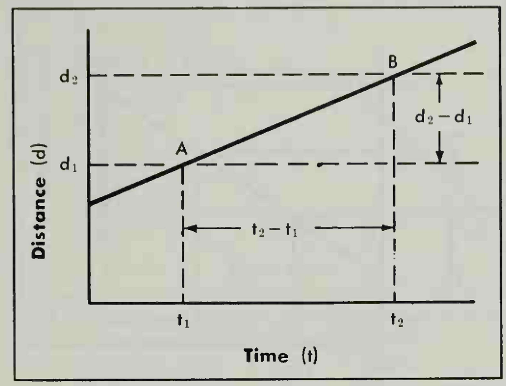
5–10. Higher speeds give steeper graphs of distance vs. time.

How can we tell from Fig. 5–9 which car is going faster? The answer is simple. One curve climbs more steeply than the other. For a given time interval, the steeper curve spans a greater interval of distance. Since the car which travels the greater distance in any given time is the faster, the faster car must be the one with the most steeply sloping graph. Car A is certainly going faster than car B. (That is why it passed B.)

Fig. 5–10 again illustrates the relationship between the steepness and the speed. The solid line is drawn for a car traveling at 25 mi/hr. We



5–11. Since the distance-time graphs for the two cars are parallel, they represent the same speed even though the positions of the graphs are different.



5–12. The slope of a straight line is found by dividing $d_2 - d_1$ "up" by $t_2 - t_1$ "over."

see that it has traveled a distance of 10 mi during the first 0.40 hr. A car going 50 mi/hr travels 20 mi during the 0.40-hr interval, and is described by the steeper dashed line of the graph.

On a d versus t graph it is the steepness alone that tells the speed. Position on the graph (or on the road) does not matter. In Fig. 5–11 the curves for cars C and D are equally steep. They are parallel to each other; therefore they describe a situation in which car D neither catches up to nor drops behind car C. It always keeps 10 miles behind C. The graphs will never cross as they did when one car passed the other (Fig. 5–9). The graphs of car C and of car D are in different positions, but they correspond to exactly the same speed.

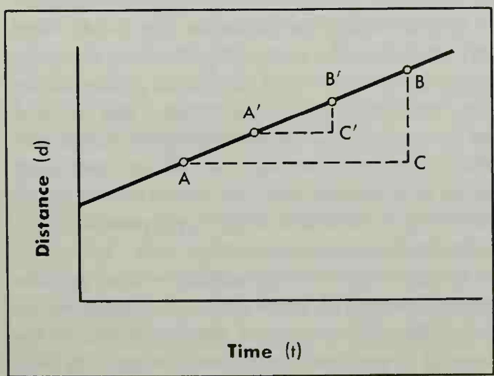
We shall now express the connection between the speed and the steepness of the d versus t graph in mathematical language. To do so we shall begin by restating the connection between speed, time, and distance.

Imagine that while driving along a turnpike you check your speedometer, using the mileposts and a watch. The watch reads 3:25:00 at the post labeled 247 miles, and 3:26:00 at 248 miles. You have traveled one mile in one minute, and your speed is a mile a minute, or 60 mi/hr. Let d_1 stand for 247 miles and d_2 for 248 miles; also let t_1 stand for the time 3:25:00 and t_2 for the time 3:26:00. In this language you can express the speed v by the equation

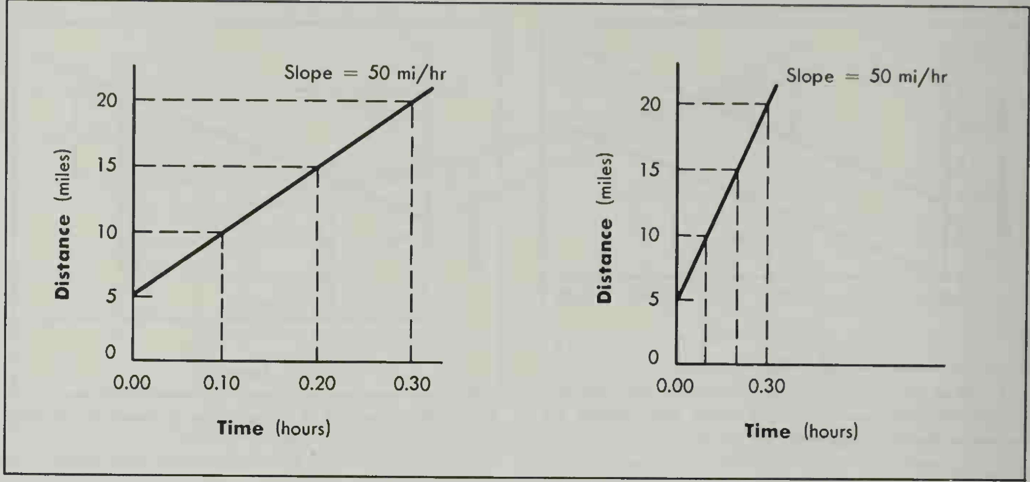
$$v = \frac{d_2 - d_1}{t_2 - t_1}.$$

In general, if we pass from position d_1 at time t_1 to position d_2 at time t_2, this equation gives the average speed with which we move.

We can now give this equation for speed a precise geometrical meaning. In Fig. 5–12, if A and B are any two points on the graph, then the vertical interval between these points is obviously $d_2 - d_1$ and the horizontal interval is $t_2 - t_1$. These two intervals, $(d_2 - d_1)$ and $(t_2 - t_1)$, completely define the steepness of the graph, for they tell how far "up" and how far "over" one point on the line is from another. As we have already discovered, the speed depends upon the steepness of the distance-time graph; and the steepness of the graph depends upon how far up the graph goes in a certain interval over. The ratio of "up" to "over" is such a useful measure of the steepness that we give it a special name. We call it the *slope* of the line. For the distance-time graph, as Fig. 5–12 shows, the slope is $d_2 - d_1$



5–13. A straight line has the same slope all along its length.



5–14. The slopes of the two graphs are the same, although their appearances differ. They will appear the same only when plotted to the same scale or on the same graph.

"up" divided by $t_2 - t_1$ "over." The slope of the distance-time graph is the speed of the car as we see from the last equation.

We determined the slope of the line in Fig. 5–12 from the ratio of "up" to "over" in going between the two points *A* and *B*. If we take any other two points such as *A'* and *B'* (Fig. 5–13), we can see from the similar triangles dotted in the figure that the ratio of "up" to "over" in the one case is exactly equal to the ratio of "up" to "over" in the other. Therefore we can use *any* pair of points on a straight line to compute its slope.

It is important to recognize that although the slope is related to the "steepness" of a graph, the angle between the graphed line and the horizontal has no particular significance since we could change the angle by replotting the data to different scale, as shown in Fig. 5–14. It is only when we are comparing lines on the same graph, as in Fig. 5–9, or on graphs plotted to the same scale, that the angle between the line and the horizontal axis helps us compare slopes. In general we must measure the vertical and horizontal intervals between two points on the graph and compute their ratio in the appropriate units — mi/hr, for instance — as in Fig. 5–14.

Our discussion of slope applies to other graphs; nothing restricts us to the graph of distance versus time. We shall shortly see the significance of the slope of a graph of *speed* versus time. In fact, we shall be dealing with slopes so often that it will be worth while to introduce a shorthand notation to indicate the process used in finding them. For instance, the slope of the distance-time graph is always equal to the ratio of an interval of distance ($d_2 - d_1$) to an interval of time ($t_2 - t_1$). Mathematicians and physicists often use the Greek letter delta, written as Δ, as an abbreviation for the phrase "an interval of." Δ is a Greek capital D chosen to stand for "difference," or "change of," or "increase of," or "an interval of." Thus, Δd means "an interval of distance" and Δt means "an interval of time." We read them "delta dee" and "delta tee."

We might compare the symbol "Δ" with some other algebraic symbol, for instance with "$\sqrt{\ }$," which means "take the square root of." In the expression $\sqrt{a}$, the symbol a stands for a number (or physical quantity) and $\sqrt{\ }$ tells us what to do with a. In a similar manner, in the expression Δt, t stands for a physical quantity, and Δ tells us what to do with t. It says "take an interval" of t or "take the difference between two values" of t.

When the ratio of two intervals is involved, as it is in determining a speed, it is customary to write the ratio as a fraction, with the understanding that the interval in the numerator takes place during the time interval of the denominator. Thus

$$v = \frac{\Delta d}{\Delta t},$$

which is read "vee equals delta dee over delta tee," means "to find the speed, take the interval of distance traveled in the time interval Δt and divide that distance interval by that time interval."

In general, when we write $\frac{\Delta a}{\Delta b}$, we always mean that we shall use the interval of a that corresponds to a given interval of b. After all, we are interested only in the ratios of related intervals. Please note that there is no sense in separating the Δ from the a or b. The whole symbol Δa has a special meaning: an interval of a. It does not mean Δ multiplied by a.

5–4. Speeds and Directions

We have learned that the speed $v = \frac{\Delta d}{\Delta t}$ is given by the slope of the distance-time graph. In Fig. 5–15 we have the distance-time graph of a complete trip made by a car. Let us interpret it by calculating the three slopes of the graph.

During the first 0.20 hr the car was traveling at constant speed, as is indicated by the constant slope. The speed may be found by taking the ratio of the distance covered and the time elapsed at any point within the interval, for example

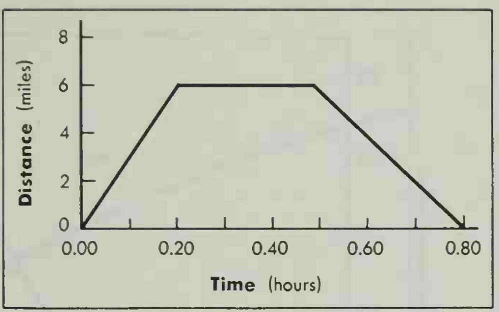
$$v_1 = \frac{\Delta d}{\Delta t} = \frac{6.0\text{ mi} - 0.0\text{ mi}}{0.20\text{ hr} - 0.00\text{ hr}} = 30\text{ mi/hr}.$$

What was the car doing from 0.20 hr until 0.50 hr? Here the slope of the graph is zero — the car was stopped. From 0.50 hr until 0.80 hr the slope is

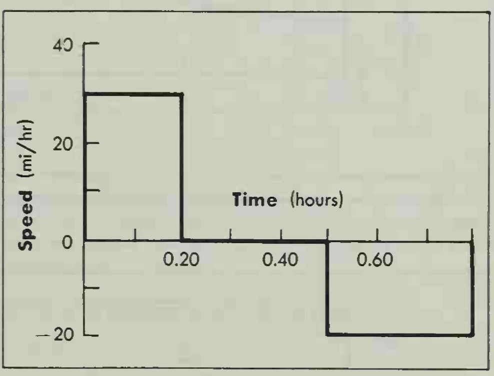
$$v_2 = \frac{\Delta d}{\Delta t} = \frac{0.0\text{ mi} - 6.0\text{ mi}}{0.80\text{ hr} - 0.50\text{ hr}} = -20\text{ mi/hr}.$$

Notice that the result is negative. The distance (0.0 mi) at the later time (0.80 hr) is less than the distance (6.0 mi) at the earlier time (0.50 hr). The minus sign just tells us that the car was traveling along the road in the opposite direction from that taken at the start of the trip. In fact, the car returned to its starting point, $d = 0$, arriving there 0.80 hr after it left. The graph indicates at a glance both the approximate speed and the direction of travel. We have now derived from it the information needed to plot the graph of Fig. 5–16.

We have seen that the quantity v can be either positive or negative. The positive sign refers to one direction of motion and the negative sign to the opposite direction.



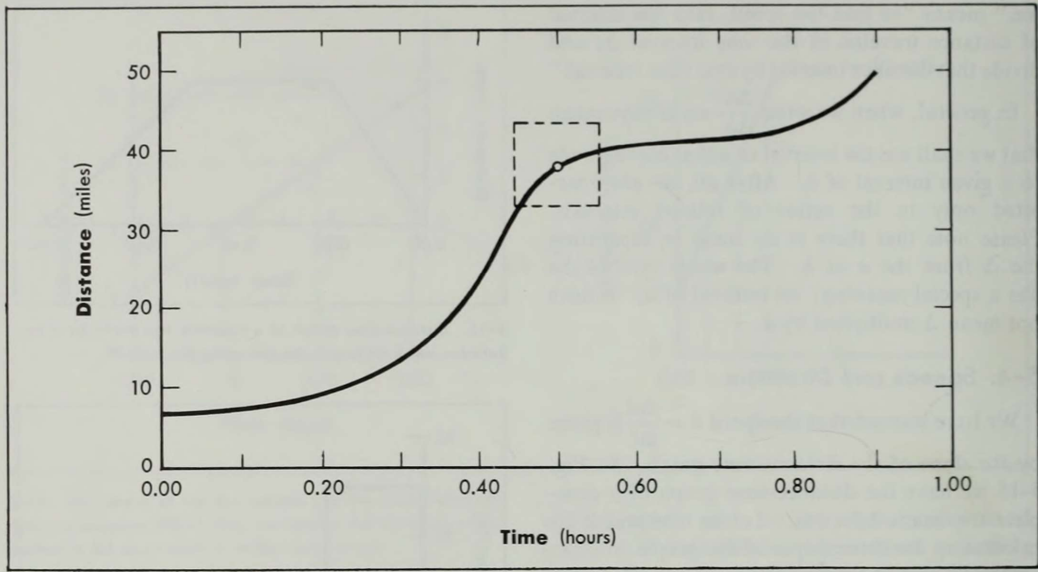
5–15. Distance-time graph of a complete trip made by a car. Between which times was the car going the fastest?



5–16. Speed-time graph of the motion shown in Fig. 5–15. The portion of the graph below the horizontal axis indicates that the car is traveling opposite to its original direction.

5–5. Instantaneous Speed — The Slope of the Tangent Line

We have been thinking of trips in which the speed is constant either throughout the trip or for different portions of the trip. The distance-time graphs were therefore made of straight lines. Fig. 5–17 is a distance-time graph of a car with continually changing speed. How can we find the speed of the car at any particular time? Here there are no straight lines, and it is not obvious how to apply the methods we have been discussing. On the other hand, if we were riding in the car, the speedometer would be able to tell us our speed. So let us think of the speedometer reading at a definite time, for instance 0.50 hr after we started our trip. How do we calculate it from the graph of Fig. 5–17?



5–17. Distance-time graph for a car with continually changing speed.